Name Harish Menon, ID b00087415

Exam One, MTH 213 , Summer 2022

Ayman Badawi

Score = $\frac{43}{43}$

QUESTION 1. (i) (5 points.) Use the 4th method and prove that $\sqrt{21}$ is irrational.

Proof by contradiction

Assume $\sqrt{21}$ is rational. Hence $\sqrt{21} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b)=1$, a, b are odd.

As such, $a = 2m+1$, $b = 2n+1$ where $m, n \in \mathbb{Z}$

$\sqrt{21} = \frac{2m+1}{2n+1}$

$21 = \frac{(2m+1)^2}{(2n+1)^2}$

$21(2n+1)(2n+1) = (2m+1)(2m+1)$

$21(4n^2+4n+1) = 4m^2+4m+1$

$84n^2+84n+21 = 4m^2+4m+1$

$84n^2+84n+20 = 4m^2+4m \quad (\div 4)$

$\underbrace{21n^2+21n+5}_{\text{Odd}} = \underbrace{m^2+m}_{\text{Even}} \Rightarrow$ a contradiction $\therefore \sqrt{21}$ is irrational

(ii) (3 points) Let x be a rational number and y be an irrational number. Prove that that xy is irrational

Proof by contradiction

Assume xy is rational. Hence $xy = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b)=1$

Since x is rational, $x = \frac{c}{d}$ where $c, d \in \mathbb{Z}$, $d \neq 0$, $\gcd(c,d)=1$

$\frac{c}{d}(y) = \frac{a}{b}$

$y = \frac{ad}{bc}$

y is rational

a contradiction

$\therefore xy$ is irrational.

(iii) (5 points) Prove that $(\sqrt{3}+\sqrt{7})^2$ is an irrational number [Hint: Use (i) and (ii) and a class notes result].

Proof by contradiction

$(\sqrt{3}+\sqrt{7})^2 = (\sqrt{3}+\sqrt{7})(\sqrt{3}+\sqrt{7}) = 3 + 2\sqrt{21} + 7$

~~Proving~~ $2\sqrt{21}$ is irrational by (i) and (ii)

~~Assume $2\sqrt{21}$ is rational. Hence $2\sqrt{21} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b)=1$~~

~~As such, $a = 2m+1$, $b = 2n+1$ where $m, n \in \mathbb{Z}$~~

~~(continued next page) Hence $2\sqrt{21} + 10$ is~~ by class notes irr + rational = Irr.

QUESTION 2. (i) (3 points) Convince me that there are 869 consecutive non-prime positive integers.

Proof directly

Let $n = 870! + 2$

Let a sequence $S = \{870!+2,\ 870!+3, \dots,\ 870!+870\}$

Consecutive numbers $= 870 - 2 + 1$
$= 869$
↳ As such there are 869 consecutive non-prime integers

(ii) (5 points) Let n be an even integer such that $8 \nmid n^2$ (i.e., 8 is not a factor of n^2). Convince me that $n^2 \pmod 8 = 4$.[Hint : Direct proof method will be easier]

proof directly

Since n is even, $n = 2k$ where $k \in \mathbb{Z}$.

$n^2 = (2k)^2 = 4k^2$

k is odd since $8 \nmid 4(\text{even})^2 \Rightarrow k = 2m+1$ where $m \in \mathbb{Z}$

$4(2m+1)^2 \bmod 8 = 4(2m+1)(2m+1) \bmod 8$
$= 4(4m^2 + 4m + 1) \bmod 8 = (16m^2 + 16m + 4) \bmod 8 \equiv 4$

For $m \in \mathbb{Z}$, $(16m^2 + 16m + 4) \bmod 8 = 4$
$\therefore n^2 \pmod 8 = 4$

(iii) (3 points) Find $gcd(64, 76)$

$76 \div 64 = 1$, $76 - 64 = 12$

$64 \div 12 = 5$, $64 - 60 = 4$

$12 \div 4 = 3$, $12 - 12 = 0$

$gcd(64, 76) = 4$

(iv) (4 points) Find two integers n, m such that $gcd(64, 76) = 64n + 76m$

$4 = 64 - 12 \times 5$
$= 64 - 5(76 - 64 \times 1)$
$= 64 - 5 \times 76 + 64 \times 5$
$= -5 \times 76 + 64 \times 6$
$= 64 \times 6 + 76 \times -5 \Rightarrow n = 6, m = -5$

QUESTION 3. (i) (4 points) If possible find the solution set to the equation $6x = 3$ over the planet Z_{15}.

$6x = 3$

$a = 6,\ r = 3,\ n = 15$

$k = \gcd(6, 15) = 3$

$3 | 3 \therefore 3$ solutions

$d = \frac{15}{3} = 5$ $\quad 6x \pmod{15} = 3$

$x = 3$

Solution set $= \{3, 8, 13\}$

(ii) (4 points, show the work). Find 7^{160003} $(mod\ 60)$

$n = 60$

$60 = 10 \times 6$

$= 2 \times 5 \times 2 \times 3$

$= 2^2 \times 5 \times 3$

$\phi(60) = 1 \times 2^1 \times 4 \times 5^0 \times 2 \times 3^0$

$= 16$

$\gcd(7, 60) = 1 \Rightarrow 7^{16} \pmod{60} = 1$

Compare 160003 and 16:

$160003 = 16(10000) + 3$

$7^{160003} = 7^{16 \times 10000} \times 7^3$

$(7^{16 \times 10000} \times 7^3) \bmod 60$

$= [(7^{16})^{10000} \bmod 60 \times 7^3 \pmod{60}]$

$= 1 \times 43$

$= 43$

(iii) (6 points) Use the CRT and find the least positive integer x such that $x\ (mod\ 7) = 3$, $x\ (mod\ 8) = 3$ and $x\ (mod\ 5) = 2$

$r_1 = 3 \quad m_1 = 7$

$r_2 = 3 \quad m_2 = 8$

$r_3 = 2 \quad m_3 = 5$

$m = 7 \times 8 \times 5 = 280$

$d_1 = \frac{280}{7} = 40$

$d_2 = \frac{280}{8} = 35$

$d_3 = \frac{280}{5} = 56$

gcd (between m's) $= 1 \therefore$ CRT can be used

Planet Z_7:

$40x = 1$

$40x \pmod 7 = 1$

$x = 3,\ c_1 = 3$

Planet Z_8:

$35x = 1$

$35x \pmod 8 = 1$

$x = 3,\ c_2 = 3$

Planet Z_5:

$56x = 1$

$56x \pmod 5 = 1$

$x = 1,\ c_3 = 1$

$x = (d_1 c_1 r_1 + d_2 c_2 r_2 + d_3 c_3 r_3) \bmod m$

$= (40 \times 3 \times 3 + 35 \times 3 \times 3 + 56 \times 1 \times 2) \bmod 280$

$= 787 \pmod{280} = 227$

(iv) (1 point) Find the maximum negative integer x that satisfies the above conditions.

$x = 227 + 280(-1) = -53$